APPENDIX B

DERIVATION OF EQUATIONS FOR FLEXURAL AND AXIAL LOADS

B-1. General

Derivations of the design equations given in paragraphs 4-2 through 4-4 are presented below. The design equations provide a general procedure that may be used to design members for combined flexural and axial load.

B-2. Axial Compression and Flexure

a. Balanced Condition

From Figure B-1, the balanced condition, Equations 4-3 and 4-4 can be derived as follows:

From equilibrium,

$$\frac{P_u}{\phi} = 0.85\ f_c'\ b\ k_u\ d - A_s f_s \tag{B-1}$$

let

$$j_u = d - \frac{a}{2} = d - \frac{k_u d}{2} \tag{B-2}$$

from moment equilibrium,

$$\frac{P_u e'}{\phi} = (0.85\ f_c'\ b\ k_u\ d)\ (j_u d) \tag{B-3}$$

Rewrite Equation B-3 as:

$$\begin{aligned} \frac{P_u e'}{\phi} &= (0.85\ f_c'\ b\ k_u\ d)\left(d - \frac{k_u d}{2}\right) \\ &= (0.85\ f_c'\ bd^2)\left(k_u - \frac{k_u^2}{2}\right) \\ &= 0.425\ f_c'\ (2k_u - k_u^2)\ bd^2 \end{aligned} \tag{B-4}$$

From the strain diagram at balanced condition (Figure B-1):

$$\frac{c_b}{d} = \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_y}$$

$$\frac{\left(\frac{k_b d}{\beta_1}\right)}{d} = \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_y} \qquad \text{(B-5)}$$

since $\varepsilon_y = \frac{f_y}{E_s}$

$$k_b = \frac{\beta_1 E_s \varepsilon_c}{E_s \varepsilon_c + f_y} \qquad \text{(B-6, Eq. 4-4)}$$

$$\text{since } e_b' = \frac{P_b e'}{P_b} \qquad \text{(B-7)}$$

e_b' is obtained by substituting Equations B-4 and B-1 into Equation B-7 with $k_u = k_b$, $f_s = f_y$ and $P_u = P_b$.

$$e_b' = \frac{0.425 f_c'(2k_b - k_b^2)\, bd^2}{0.85 f_c'\, k_b bd - f_y \rho bd} \qquad \text{(B-8)}$$

Therefore

$$\frac{e_b'}{d} = \frac{2k_b - k_b^2}{2k_b - \frac{f_y \rho}{0.425 f_c'}} \qquad \text{(B-9, Eq. 4-3)}$$

b. Sections Controlled by Tension (Figure B-1).

ϕP_n is obtained from Equation B-1 with $f_s = f_y$ as:

$$\phi P_n = \phi\,(0.85\ f_c'\ b\ k_u d - A_s f_y) \qquad \text{(B-10, Eq. 4-5)}$$

$$\phi P_n = \phi\,(0.85\ f_c'\ k_u - \rho f_y)\ bd$$

The design moment ϕM_n is expressed as:

$$\phi M_n = \phi P_n e$$

$$\phi M_n = \phi P_n \left[\frac{e'}{d} - \left(1 - \frac{h}{2d}\right)\right] d \qquad \text{(B-11)}$$

Therefore,

$$\phi M_n = \phi(0.85\ f_c'\ k_u - f_y\rho)\left[\frac{e'}{d} - \left(1 - \frac{h}{2d}\right)\right] bd^2 \qquad \text{(B-12, Eq. 4-6)}$$

Substituting Equation B-1 with $f_s = f_y$ into Equation B-4 gives

$$(0.85\ f_c'\ k_u\ bd - f_y\rho bd)e' = 0.425 f_c'\ (2k_u - k_u^2)\ bd^2 \qquad \text{(B-13)}$$

which reduces to

$$k_u^2 + 2\left(\frac{e'}{d} - 1\right)\ k_u - \frac{f_y\rho e'}{0.425 f_c' d} = 0 \qquad \text{(B-14)}$$

Solving by the quadratic equation:

$$k_u = \sqrt{\left(\frac{e'}{d} - 1\right)^2 + \left(\frac{\rho f_y}{0.425 f_c'}\right)\frac{e'}{d}} - \left(\frac{e'}{d} - 1\right) \qquad \text{(B-15, Eq. 4-7)}$$

c. Sections Controlled by Compression (Figure B-1)

ϕP_n is obtained from Equation B-1

$$\phi P_n = \phi \left(0.85\ f_c'\ k_u - \rho f_s\right) bd \qquad \text{(B-16, Eq. 4-8)}$$

and ϕM_n is obtained by multiplying Equation B-16 by e.

$$\phi M_n = \phi \left(0.85\ f_c'\ k_u - \rho f_s\right) \left[\frac{e'}{d} - \left(1 - \frac{h}{2d}\right)\right] bd^2 \qquad \text{(B-17)}$$

The steel stress, f_s, is expressed as $f_s = E_s \varepsilon_s$.

From Figure B-1.

$$\frac{c}{d} = \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_s}$$

or

$$\frac{\left(\frac{k_u d}{\beta_1}\right)}{d} = \frac{\varepsilon_c}{\varepsilon_c + \varepsilon_s}$$

Therefore,

$$f_s = \frac{E_s \varepsilon_c (\beta_1 - k_u)}{k_u} \qquad \text{(B-18, Eq. 4-10)}$$

Substituting Equations B-1 and B-18 into B-4 gives

$$0.85\ f_c'\ k_u\ bde' - \left[\frac{E_s\ \varepsilon_c (\beta_1 - k_u)}{k_u}\right] \rho bde' = 0.425\ f_c' (2k_u - k_u^2) bd^2 \qquad \text{(B-19)}$$

which can be arranged as

$$k_u^3 + 2\left(\frac{e'}{d} - 1\right) k_u^2 + \left(\frac{E_s \varepsilon_c \rho e'}{0.425 f_c' d}\right) k_u - \frac{\beta_1 E_s \varepsilon_c \rho e'}{0.425 f_c' d} = 0 \qquad \text{(B-20, Eq. 4-11)}$$

B-3. Flexural and Compressive Capacity-Tension and Compression Reinforcement (Figure B-2)

a. Balanced Condition

Using Figure B-2, the balanced condition, Equation 4-13 can be derived as follows:

From equilibrium,

$$\frac{P_u}{\phi} = 0.85\ f_c'\ k_u bd + f_s'\ \rho' bd - f_s \rho bd \qquad \text{(B-21)}$$

In a manner similar to the derivation of Equation B-4, moment equilibrium results in

$$\frac{P_u e'}{\phi} = 0.425\ f_c' \left(2k_u - k_u^2\right) bd^2 + f_s'\ \rho' bd(d - d') \qquad \text{(B-22)}$$

As in Equation B-6,

$$k_b = \frac{\beta_1 E_s \varepsilon_c}{E_s \varepsilon_c + f_y} \qquad \text{(B-23)}$$

$$\text{since } e_b' = \frac{P_b e'}{P_s} \qquad \text{(B-24)}$$

and using Equations B-21 and B-22:

$$e_b' = \frac{0.425\ f_c'\ (2k_b - k_b^2) bd^2 + f_s'\ \rho' bd(d - d')}{0.85\ f_c'\ k_b bd + f_s'\ \rho' bd - f_s\ \rho bd} \qquad \text{(B-25)}$$

which can be rewritten as

$$e_b' = \frac{(2k_b - k_b^2)d + \dfrac{f_s'\rho'}{0.425 f_c'}(d - d')}{2k_b + \dfrac{f_s\rho'}{0.425 f_c'} - \dfrac{f_y\rho}{0.425 f_c'}}$$

or

$$\frac{e_b'}{d} = \frac{2k_b - k_b^2 + \dfrac{f_s'\rho'\left(1 - \dfrac{d'}{d}\right)}{0.425 f_c'}}{2k_b - \dfrac{f_y\rho}{0.425 f_c'} + \dfrac{f_s'\rho'}{0.425 f_c'}} \qquad \text{(B-26, Eq. 4-13)}$$

b. Sections Controlled by Tension (Figure B-2)

ϕP_n is obtained as Equation B-21 with $f_s = f_y$.

$$\phi P_n = \phi\,(\,0.85\ f_c'\ k_u + \rho' f_s' - \rho f_y\,)\ bd \qquad \text{(B-27, Eq. 4-14)}$$

Using Equations B-11 and B-27,

$$\phi M_n = \phi\,(0.85\ f_c'\ k_u + \rho' f_s' - \rho f_y)\left[\frac{e'}{d} - \left(1 - \frac{h}{2d}\right)\right] bd^2 \qquad \text{(B-28, Eq. 4-15)}$$

From Figure B-2

$$\frac{\varepsilon_s'}{c - d'} = \frac{\varepsilon_y}{d - c}\ ;\ f_s' = E_s\varepsilon_s'\ ;\ c = \frac{k_u d}{\beta_1}$$

Therefore,

$$\frac{f_s'}{E_s} = \left(\frac{k_u d}{\beta_1} - d'\right)\left(\frac{\varepsilon_y}{d - \frac{k_u d}{\beta_1}}\right)$$

or

$$f_s' = \frac{\left(k_u - \beta_1 \frac{d'}{d}\right)}{(\beta_1 - k_u)} E_s \varepsilon_y \qquad \text{(B-29, Eq. 4-16)}$$

Substituting Equation B-21 with $f_s = f_y$ into Equation B-22 gives,

$$\begin{aligned}&(0.85\, f_c'\, k_u bd + f_s' \rho' bd - f_y \rho bd) e' \\ &\quad = 0.425\, f_c' \left(2k_u - k_u^2\right) bd^2 + f_s' \rho' bd(d - d')\end{aligned} \qquad \text{(B-30)}$$

Using Equation B-29, Equation B-30 can be written as:

$$\begin{aligned}&k_u^3 + \left[2\left(\frac{e'}{d} - 1\right) - \beta_1\right] k_u^2 - \left\{\frac{f_y}{0.425 f_c'}\left[\rho'\left(\frac{e'}{d} + \frac{d'}{d} - 1\right) + \frac{\rho e'}{d}\right]\right. \\ &\quad \left. + 2\,\beta_1\left(\frac{e'}{d} - 1\right)\right\} k_u + \frac{f_y \beta_1}{0.425 f_c'}\left[\rho' \frac{d'}{d}\left(\frac{e'}{d} + \frac{d'}{d} - 1\right) + \frac{\rho e'}{d}\right] \\ &\quad = 0\end{aligned} \qquad \text{(B-31, Eq. 4-17)}$$

c. Sections Controlled by Compression (Figures B-2)

ϕP_n is obtained from equilibrium

$$\phi P_n = \phi\,(0.85\, f_c'\, k_u + \rho' f_s' - \rho f_s)\, bd \qquad \text{(B-32, Eq. 4-18)}$$

Using Equations B-11 and B-32,

$$\phi M_n = \phi \left(0.85\ f_c'\ k_u + \rho' f_s' - \rho f_s\right) \left[\frac{e'}{d} - \left(1 - \frac{h}{2d}\right)\right] bd^2 \qquad \text{(B-33, Eq. 4-19)}$$

From Figure B-2

$$\frac{\varepsilon_s}{d - c} = \frac{\varepsilon_c}{c}\ ;\quad f_s = E_s \varepsilon_s\ ;\quad c = \frac{k_u d}{\beta_1}$$

which can be written as

$$f_s = \frac{E_s \varepsilon_c (\beta_1 - k_u)}{k_u} \qquad \text{(B-34, Eq. 4-20)}$$

Also,

$$\frac{\varepsilon_s'}{c - d'} = \frac{\varepsilon_c}{c}$$

which can be rewritten as

$$f_s' = \frac{E_s \varepsilon_c \left[k_u - \beta_1 \left(\frac{d'}{d}\right)\right]}{k_u} \qquad \text{(B-35, Eq. 4-21)}$$

From Equations B-21 and B-22

$$\begin{aligned} &\left(0.85 f_c'\ k_u bd = f_s' \rho' bd - f_s \rho bd\right) e' \\ &\quad = 0.425\ f_c'\ \left(2k_u - k_u^2\right) bd^2 + f_s'\ \rho' bd(d - d') \end{aligned} \qquad \text{(B-36)}$$

Substituting Equations B-34 and B-35 with $k_b = k_u$ into Equation B-36 gives

$$k_u^3 + 2\left(\frac{e'}{d} - 1\right)k_u^2 + \frac{E_s\varepsilon_c}{0.425f_c'}\left[(\rho + \rho')\left(\frac{e'}{d}\right) - \rho'\left(1 - \frac{d'}{d}\right)\right]k_u$$

$$- \frac{\beta_1 E_s\varepsilon_c}{0.425f_c'}\left[\rho'\left(\frac{d'}{d}\right)\left(\frac{e'}{d} + \frac{d'}{d} - 1\right) + \rho\left(\frac{e'}{d}\right)\right] = 0 \qquad \text{(B-37, Eq. 4-22)}$$

B-4. Flexural and Tensile Capacity

a. Pure Tension (Figure B-3)

From equilibrium (double reinforcement)

$$\phi P_n = \phi(A_s + A_s')f_y \qquad \text{(B-38)}$$

For design, the axial load strength of tension members is limited to 80 percent of the design axial load strength at zero eccentricity.

Therefore,

$$\phi P_{n(\max)} = 0.8\phi(\rho + \rho')f_y\, bd \qquad \text{(B-39, Eq. 4-23)}$$

b. For the case where $1 - \frac{h}{2d} \geq \frac{e'}{d} \geq 0$, the applied tensile resultant $\frac{P_u}{\phi}$ lies between the two layers of steel.

From equilibrium

$$\phi P_n = \phi\,(A_s f_y + A_s' f_s')$$

or

$$\phi P_n = \phi\,(\rho f_y + \rho' f_s')\,bd \qquad \text{(B-40, Eq. 4-24)}$$

and

$$\phi M_n = \phi P_n\left[\left(1 - \frac{h}{2d}\right) - \frac{e'}{d}\right]d$$

or

$$\phi M_n = \phi\left(\rho f_y + \rho' f_s'\right)\left[\left(1 - \frac{h}{2d}\right) - \frac{e'}{d}\right] bd^2 \qquad \text{(B-41, Eq. 4-25)}$$

From Figure B-3,

$$\frac{\varepsilon_s'}{a + d'} = \frac{\varepsilon_y}{a + d}$$

which can be rewritten as

$$f_s' = f_y \frac{\left(k_u + \frac{d'}{d}\right)}{\left(k_u + 1\right)} \qquad \text{(B-42, Eq. 4-26)}$$

From Figure B-3 equilibrium requires:

$$A_s f_s e' = A_s' f_s' \left(d - d' - e'\right) \qquad \text{(B-43)}$$

Substituting Equation B-42 and $f_s = f_y$ into Equation B-43 results in

$$k_u = \frac{\rho'\left(\frac{d'}{d}\right)\left(1 - \frac{d'}{d} - \frac{e'}{d}\right) - \rho\left(\frac{e'}{d}\right)}{\rho\left(\frac{e'}{d}\right) - \rho'\left(1 - \frac{d'}{d} - \frac{e'}{d}\right)} \qquad \text{(B-44, Eq. 4-27)}$$

c. The case where $(e'/d) < 0$ is similar to the combined flexural and compression case. Therefore, k_u is derived in a manner similar to the derivation of Equation B-15 and is given as

$$k_u = -\left(\frac{e'}{d} - 1\right) - \sqrt{\left(\frac{e'}{d} - 1\right)^2 + \left(\frac{\rho f_y}{0.425 f_c'}\right)\frac{e'}{d}} \qquad \text{(B-45, Eq. 4-28)}$$

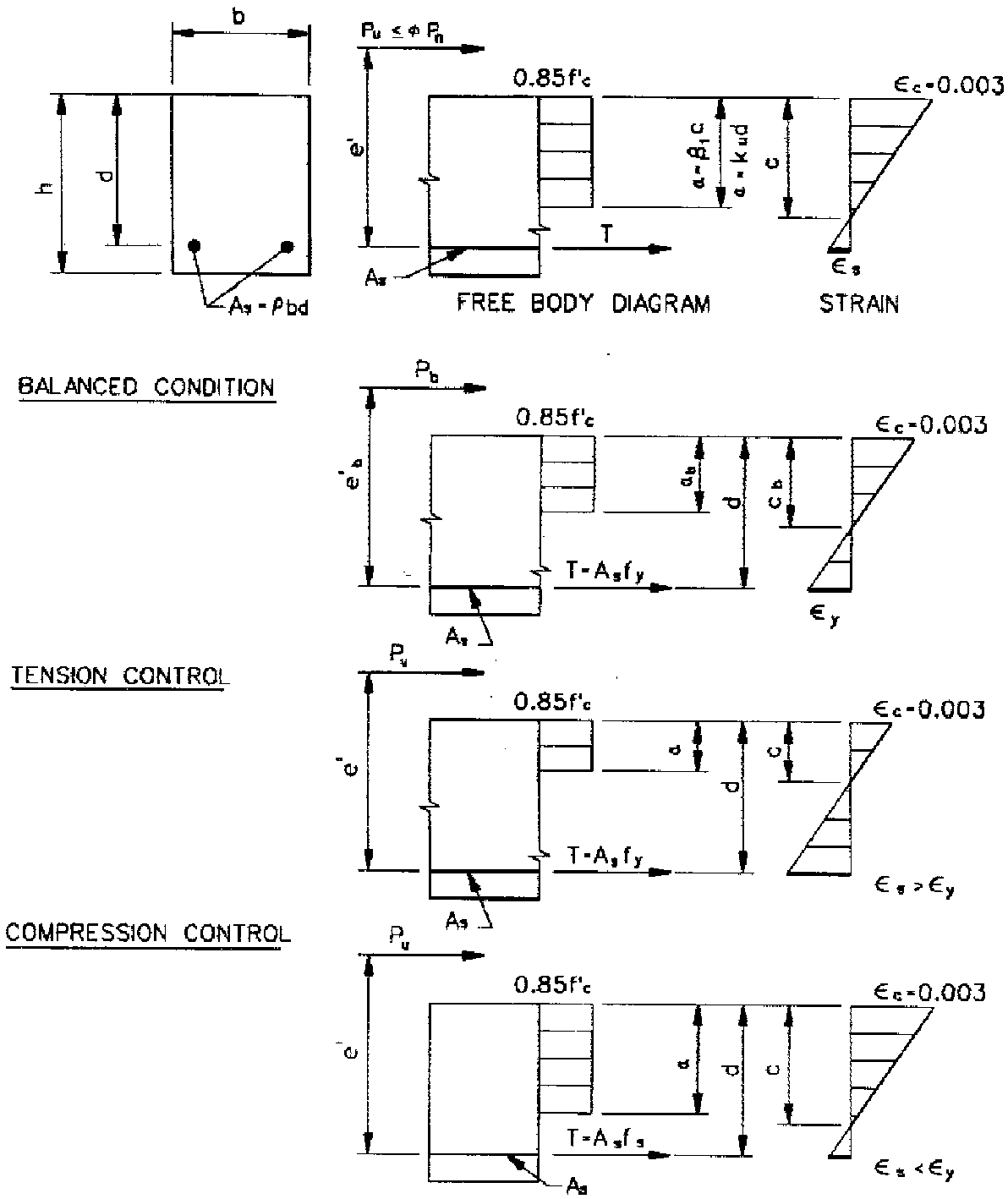

Figure B-1. Axial compression and flexure, single reinforcement

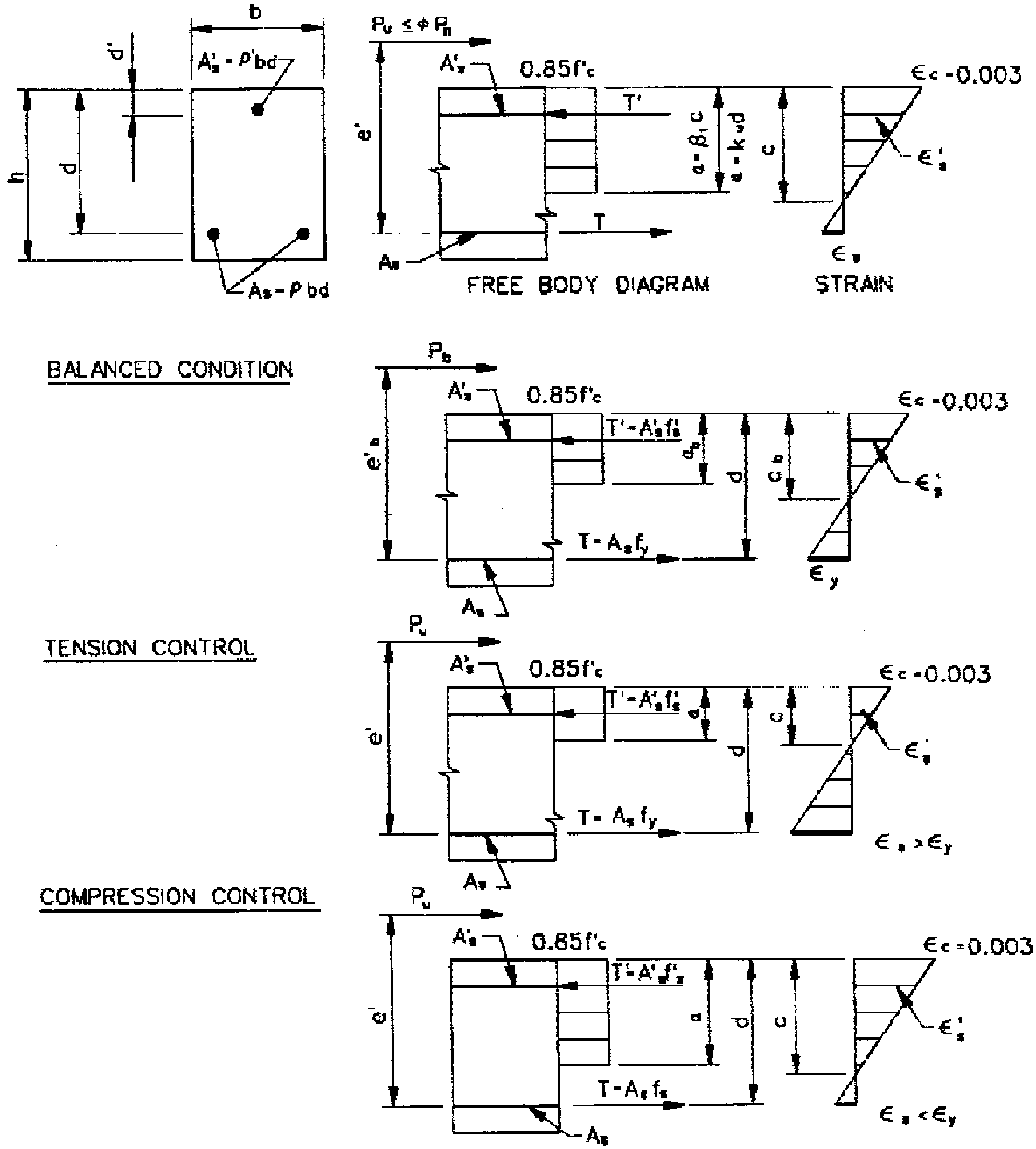

Figure B-2. Axial compression and flexure, double reinforcement

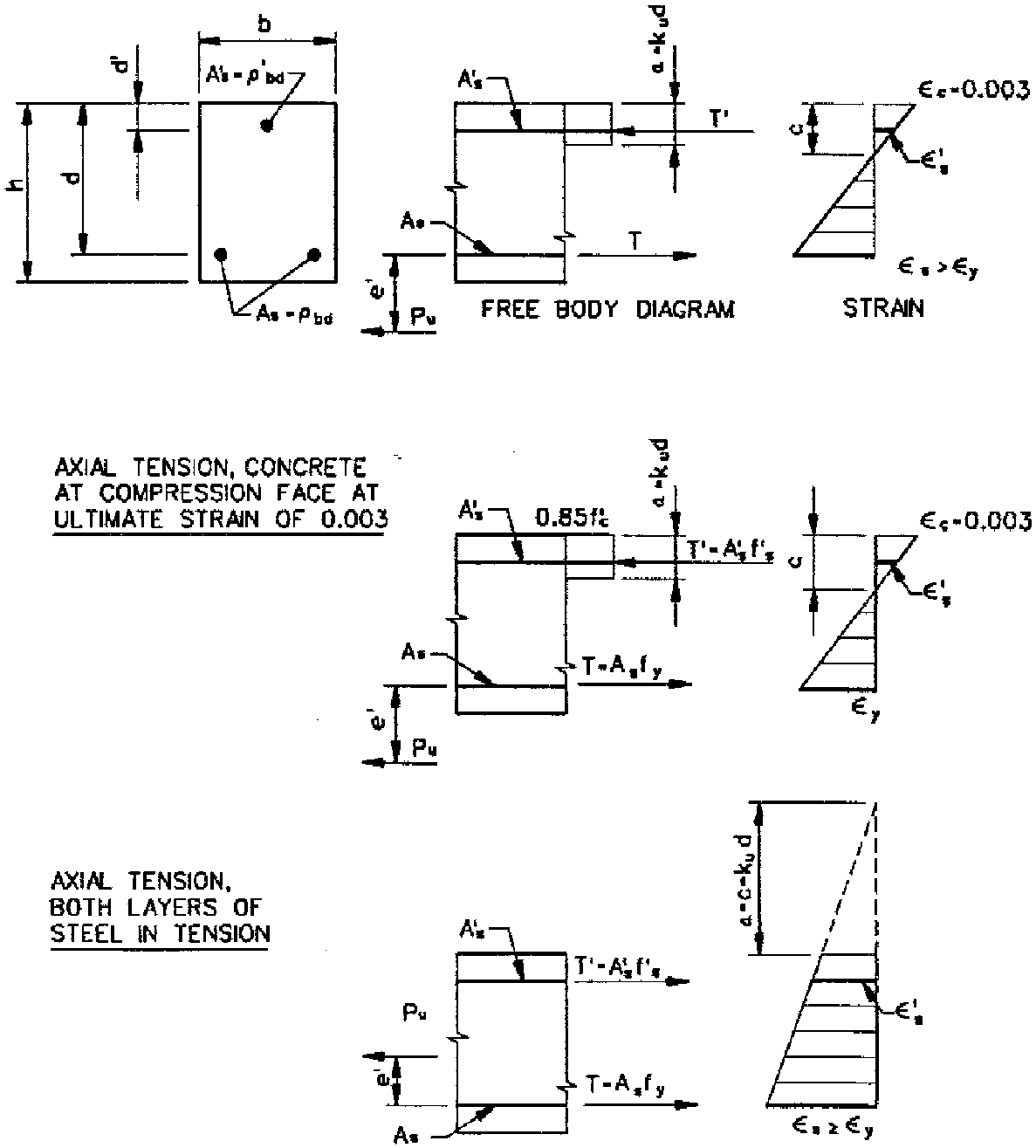

Figure B-3. Axial tension and flexure, double reinforcement