Appendix E
Wedge Equations

1. General Wedge Equation.

a. Nomenclature

α_i = angle between failure plane and horizontal

$Ø_i$ = internal friction angle of material in *ith* wedge. Earth and rock wedges only.

c_i = cohesive strength of material in *ith* wedge. Earth and rock wedges only.

W_i = weight of material in *ith* wedge.

V_i = surcharge load acting on *ith* wedge.

H_{Li} = horizontal force on *ith* wedge, acting to the right.

H_{Ri} = horizontal force on *ith* wedge, acting to left.

U_i = uplift (water) load acting normal to failure plane.

N_i = force acting normal to failure plane of *ith* wedge.

T_i = shear force acting parallel to failure plane of *ith* wedge.

P_i = horizontal force due to the *ith* wedge.

L_i = length of wedge base.

b. Sign Convention. The x and y axes are to the right and upward, respectively. The t and n axes are tangent and normal to the failure plane. Where the failure plane is not horizontal, the angle from the x-axis to the t -axis is determined by the right hand rule.

c. Equilibrium equations.

$\Sigma F_n = 0$

$$0 = N_i + U_i - W_i \cos\alpha_i - V_i \cos\alpha_i - H_{Li} \sin\alpha_i + H_{Ri} \sin\alpha_i - (P_{i-1} - P_i) \sin\alpha_i$$
$$N_i = (W_i + V_i) \cos\alpha_i - U_i + (H_{Li} - H_{Ri}) \sin\alpha_i + (P_{i-1} - P_i) \sin\alpha_i$$

$\Sigma F_t = 0$

$$0 = -T_i - W_i \sin\alpha_i - V_i \sin\alpha_i + H_{Li} \cos\alpha_i - H_{Ri} \cos\alpha_i + (P_{i-1} - P_i) \cos\alpha_i$$
$$T_i = (H_{Li} - H_{Ri}) \cos\alpha_i - (W_i + V_i) \sin\alpha_i + (P_{i-1} - P_i) \cos\alpha_i$$

d. Mohr-Coulomb failure criterion.

$$T_F = N_i \tan\phi_i + c_i L_i$$

e. Sliding factor of safety definition.

$$FS_i = \frac{T_F}{T_i} = \frac{N_i \tan\phi_i + c_i L_i}{T_i}$$

f. Governing wedge equation.

$$FS_i = \frac{\left[(W_i + V_i)\cos\alpha_i - U_i + (H_{Li} - H_{Ri})\sin\alpha_i + (P_{i-1} - P_i)\sin\alpha_i\right]\tan\phi_i + c_i L_i}{(H_{Li} - H_{Ri})\cos\alpha_i - (W_i + V_i)\sin\alpha_i + (P_{i-1} - P_i)\cos\alpha_i} \tag{E-1}$$

$$(P_{i-1} - P_i) = \frac{\left[(W_i + V_i)\cos\alpha_i - U_i + (H_{Li} - H_{Ri})\sin\alpha_i\right]\frac{\tan\phi_i}{FS_i} - (H_{Li} - H_{Ri})\cos\alpha_i + (W_i + V_i)\sin\alpha_i + \frac{c_i L_i}{FS_i}}{\cos\alpha_i - \frac{\tan\phi_i \sin\alpha_i}{FS_i}}$$

A negative value of the difference (P_{i-1} - P_i) indicates that the applied forces acting on the ith wedge exceed the forces resisting sliding along the base of the wedge. A positive value of the difference (P_{i-1} - P_i) indicates that the applied forces acting on the ith wedge are less than the forces resisting sliding along the base of that wedge. The governing equation for (P_{i-1} - P_i) applies to the individual wedges. For the system of wedges to act as an integral failure mechanism, the safety factors of all wedges must be identical.

$$FS_1 = FS_2 = \ldots\ldots FS_{i-1} = FS_i = FS_{i+1} = \ldots\ldots FS_N$$

where N = the number of wedges in the failure mechanism.

The actual safety factor for sliding equilibrium is determined by satisfying overall horizontal equilibrium ($\Sigma F_H = 0$) for the entire system of wedges.

$$\sum_{i=1}^{N} (P_{i-1} - P_i) = 0 \qquad \text{and} \qquad P_0 \equiv 0 \qquad P_N \equiv 0$$

Usually, an iterative solution process is used to determine the actual safety factor for sliding equilibrium. An example of a typical static loading condition analysis for a multiple-wedge system is presented in Example D2 of Appendix D. Note that if $\Sigma F_H < 0$, the factor of safety is less than the trial factor of safety, and if $\Sigma F_H > 0$, the factor of safety is greater than the trial factor of safety.

2. Critical Slip Angle for Driving-Side Wedge with Wall Friction on the Vertical Face

This section illustrates derivation of the critical slip angle for the limit equilibrium condition. Figure E-1 shows the forces acting on a driving-side wedge.

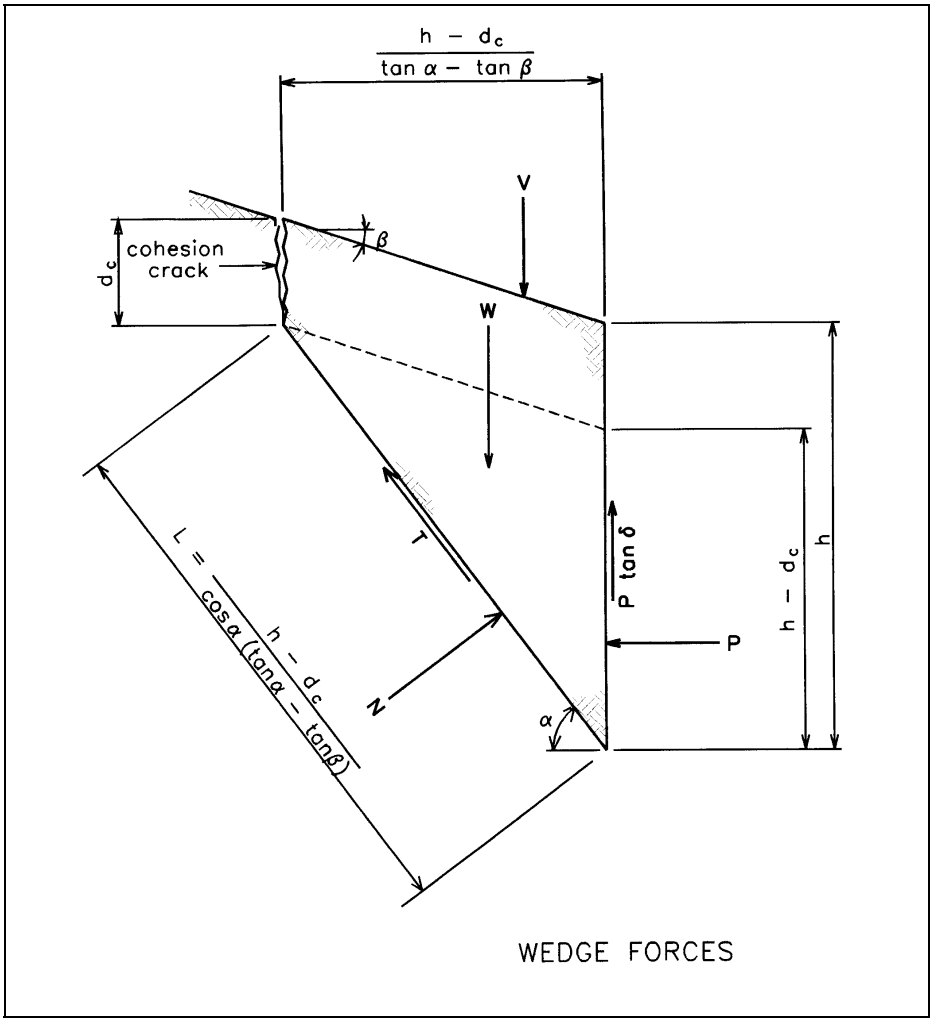

Figure E-1 Forces acting on wedge

For equilibrium to exist:

$$N = (W + V)\cos\alpha + P\sin\alpha - P\tan\delta\cos\alpha$$

$$T = (W + V)\sin\alpha - P\cos\alpha - P\tan\delta\sin\alpha$$

$$T \text{ must also} = N\tan\phi + cL$$

$$T = N\tan\phi + cL = (W + V)\tan\phi\cos\alpha + P\tan\phi\sin\alpha - P\tan\delta\tan\phi\cos\alpha + cL$$

Equating the two expressions for *T*, dividing them by cos α, and solving for *P*, the following is obtained:

$$P = \frac{(W + V)(\tan\alpha - \tan\phi) - \dfrac{cL}{\cos\alpha}}{(1 - \tan\delta\tan\phi) + (\tan\delta + \tan\phi)\tan\alpha}$$

where

W = weight of soil in wedge

V = strip surcharge

$$W = \frac{\gamma(h^2 - d_c^2)}{2(\tan\alpha - \tan\beta)}, \quad \gamma = \text{unit weight of soil}$$

$$L = \frac{h - d_c}{\cos\alpha\,(\tan\alpha - \tan\beta)}$$

substituting the above values for W and L into the equation for P:

$$P = \frac{\left[\dfrac{\gamma(h^2 - d_c^2)}{2(\tan\alpha - \tan\beta)} + V\right](\tan\alpha - \tan\phi) - \dfrac{c(h - d_c)}{\cos^2\alpha\,(\tan\alpha - \tan\beta)}}{(1 - \tan\delta\tan\phi) + (\tan\delta + \tan\phi)\tan\alpha}$$

$$\text{Note that: } \frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

divide both sides of this equation by:

$$\frac{\gamma(h^2 - d_c^2)}{2}, \quad \text{also substitute } 1 + \tan^2\alpha \text{ for } \frac{1}{\cos^2\alpha}$$

to obtain:

$$\frac{2P}{\gamma(h^2 - d_c^2)} = \frac{(\tan\alpha - \tan\phi) + \dfrac{2V}{\gamma(h^2 - d_c^2)}(\tan\alpha - \tan\beta)(\tan\alpha - \tan\phi) - \dfrac{2c(1 + \tan^2\alpha)}{\gamma(h + d_c)}}{(\tan\alpha - \tan\beta)[1 - \tan\delta\tan\phi + (\tan\delta + \tan\phi)\tan\alpha]} = \frac{m}{n}$$

combining terms the above equation becomes:

$$\frac{m}{n} = \frac{(\tan\alpha - \tan\phi) + \dfrac{2V}{\gamma(h^2 - d_c^2)}[\tan^2\alpha - (\tan\beta + \tan\phi)\tan\alpha + \tan\beta\tan\phi] - \dfrac{2c(1 + \tan^2\alpha)}{\gamma(h + d_c)}}{(\tan\delta + \tan\phi)\tan^2\alpha + [1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan\alpha - \tan\beta(1 - \tan\delta\tan\phi)}$$

The necessary condition for P to be either a maximum or a minimum is that the derivative of m/n, with respect to α, be equal to zero. The derivative of m/n is:

$$\frac{d(m/n)}{d\alpha} = \frac{n\left(\dfrac{dm}{d\alpha}\right) - m\left(\dfrac{dn}{d\alpha}\right)}{n^2} = 0$$

from this it can be seen that if both sides of the equation are multiplied by n^2 the maxima-minima condition becomes:

$$n\ \frac{dm}{d\,\alpha} - m\ \frac{dn}{d\,\alpha} = 0$$

After differentiating this equation, it will be found that all terms are multiplied by $\sec^2\alpha$ ($d\tan\alpha/d\alpha$). For simplification, both sides of the equation will be divided (at the start) by $\sec^2\alpha$ to eliminate it.

Since all of the terms in the numerator (*m*) are over a common denominator, differentiation will be done separately for each numerator term; they will be combined at the end to furnish a complete solution. The derivative $dn/d\alpha$ is constant for all terms.

$$\frac{dn}{d\,\alpha} = 2\,(\tan\delta + \tan\phi)\tan\alpha + 1 - \tan\delta\,\tan\phi - \tan\beta\,(\tan\delta + \tan\phi)$$

<u>The "*W*" term ($\tan\alpha - \tan\phi$)</u>:

$$\frac{dm}{d\,\alpha} = 1$$

$$n\frac{dm}{d\,\alpha} = (\tan\delta + \tan\phi)\tan^2\alpha + (1 - \tan\delta\,\tan\phi)\tan\alpha - \tan\beta(\tan\delta + \tan\phi)\tan\alpha - \tan\beta(1 - \tan\delta\,\tan\phi)$$

$$m\frac{dn}{d\,\alpha} = 2(\tan\delta + \tan\phi)\tan^2\alpha + (1 - \tan\delta\,\tan\phi)\tan\alpha - \tan\beta(\tan\delta + \tan\phi)\tan\alpha$$
$$- 2\tan\phi\,(\tan\delta + \tan\phi)\tan\alpha - \tan\phi(1 - \tan\delta\,\tan\phi) + \tan\beta\,\tan\phi\,(\tan\delta + \tan\phi)$$

$$n\frac{dm}{d\,\alpha} - m\frac{dn}{d\,\alpha} = -(\tan\delta + \tan\phi)\tan^2\alpha + 2\tan\phi(\tan\delta + \tan\phi)\tan\alpha + \tan\phi - \tan\beta - (\tan\delta + \tan\beta)\tan^2\phi = 0$$

<u>The "*V*" term</u>:

$$\frac{dm}{d\,\alpha} = \left[\frac{4V}{\gamma\,(h^2 - d_c^2)}\right]\tan\alpha - \left[\frac{2V}{\gamma\,(h^2 - d_c^2)}\right](\tan\beta + \tan\phi)$$

$$n\frac{d\,m}{d\,\alpha} = \left[\frac{4V}{\gamma(h^2 - d_c^2)}\right](\tan\delta + \tan\phi)\tan^3\alpha + \left[\frac{4V}{\gamma(h^2 - d_c^2)}\right][1 - \tan\delta\,\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan^2\alpha$$
$$- \left[\frac{4V}{\gamma(h^2 - d_c^2)}\right]\tan\beta(1 - \tan\delta\,\tan\phi)\tan\alpha - \left[\frac{2V}{\gamma(h^2 - d_c^2)}\right](\tan\beta + \tan\phi)(\tan\delta + \tan\phi)\tan^2\alpha$$
$$- \left[\frac{2V}{\gamma\,(h^2 - d_c^2)}\right](\tan\beta + \tan\phi)[1 - \tan\delta\,\tan\phi - \tan\beta\,(\tan\delta + \tan\phi)]\tan\alpha$$
$$+ \left[\frac{2V}{\gamma(h^2 - d_c^2}\right]\tan\beta\,(\tan\beta + \tan\phi)(1 - \tan\delta\,\tan\phi)$$

$$m\frac{dn}{d\alpha}=\left[\frac{4V}{\gamma(h^2-d_c^2)}\right][(\tan\delta+\tan\phi)\tan^3\alpha-(\tan\delta+\tan\phi)(\tan\beta+\tan\phi)\tan^2\alpha+\tan\beta\tan\phi(\tan\delta+\tan\phi)\tan\alpha]$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)][\tan^2\alpha-(\tan\beta+\tan\phi)\tan\alpha]$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]\tan\beta\tan\phi=0$$

$$n\frac{dm}{d\alpha}-m\frac{dn}{d\alpha}=\left[\frac{2V}{\gamma(h^2-d_c^2)}\right](1+\tan^2\phi)\tan^2\alpha-\left[\frac{4V}{\gamma(h^2-d_c^2)}\right]\tan\beta(1+\tan^2\phi)\tan\alpha$$
$$+\left[\frac{2V}{\gamma(h^2-d_c^2)}\right]\tan^2\beta(1+\tan^2\phi)=0$$

<u>The “c” term:</u>

$$m=-\frac{2c}{\gamma(h+d_c)}-\frac{2c\tan^2\alpha}{\gamma(h+d_c)},\quad \frac{dm}{d\alpha}=-\frac{4c\tan\alpha}{\gamma(h+d_c)}$$
$$\frac{dn}{d\alpha}=2(\tan\delta+\tan\phi)\tan\alpha+[1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]$$

$$n\frac{dm}{d\alpha}=-\left[\frac{4c}{\gamma(h+d_c)}\right](\tan\delta+\tan\phi)\tan^3\alpha$$
$$-\left[\frac{4c}{\gamma(h+d_c)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]\tan^2\alpha$$
$$+\left[\frac{4c}{\gamma(h+d_c)}\right][\tan\beta(1-\tan\delta\tan\phi)]\tan\alpha$$

$$m\frac{dn}{d\alpha}=-\left[\frac{4c}{\gamma(h+d_c)}\right](\tan\delta+\tan\phi)\tan^3\alpha$$
$$-\left[\frac{2c}{\gamma(h+d_c)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]\tan^2\alpha$$
$$-\left[\frac{4c}{\gamma(h+d_c)}\right](\tan\delta+\tan\phi)\tan\alpha$$
$$-\left[\frac{2c}{\gamma(h+d_c)}\right][1-\tan\delta\tan\phi-\tan\beta(\tan\delta+\tan\phi)]$$

$$n\frac{dm}{d\alpha} - m\frac{dn}{d\alpha} = -\left[\frac{2c}{(h+d_c)}\right][\,1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)]\tan^2\alpha$$
$$+\left[\frac{4c}{\gamma(h+d_c)}\right][\tan\beta(1 - \tan\delta\tan\phi) + \tan\delta + \tan\phi\,]\tan\alpha$$
$$+\left[\frac{2c}{\gamma(h+d_c)}\right][1 - \tan\delta\tan\phi - \tan\delta\tan\beta - \tan\beta\tan\phi\,] = 0$$

<u>Combining "*W*", "*V*", and "*c*" terms:</u>

$$-[\tan\phi + \tan\delta - \left[\frac{2V}{\gamma(h^2 - d_c^2)}\right](1+\tan^2\phi) + \left[\frac{2c}{\gamma(h+d_c)}\right](1 - \tan\delta\tan\phi - \tan\beta[\tan\delta + \tan\phi])]\tan^2\alpha$$
$$+[\,2\tan\phi(\tan\delta + \tan\phi) - \left[\frac{4V}{\gamma(h^2 - d_c^2)}\right]\tan\beta(1+\tan^2\phi)$$
$$+\left[\frac{4c}{\gamma(h+d_c)}\right](\tan\beta + \tan\phi + \tan\delta[\,1 - \tan\beta\tan\phi\,])\,]\tan\alpha$$
$$+[\tan\phi - \tan\beta - (\tan\delta + \tan\beta)\tan^2\phi + \left[\frac{2V}{\gamma(h^2 - d_c^2)}\right]\tan^2\beta(1+\tan^2\phi)$$
$$+\left[\frac{2c}{\gamma(h+d_c)}\right][\,1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi)\,] = 0$$

In order to make the above equation less cumbersome, let:

$$1 - \tan\delta\tan\phi - \tan\beta(\tan\delta + \tan\phi) = r$$
$$\tan\beta + \tan\phi + \tan\delta(1 - \tan\beta\tan\phi) = s$$
$$\tan\phi - \tan\beta - (\tan\delta + \tan\beta)\tan^2\phi = t$$

Then the equation becomes:

$$-\left[\tan\phi + \tan\delta - \left(\frac{2V}{\gamma(h^2 - d_c^2)}\right)(1+\tan^2\phi) + \left(\frac{2c}{\gamma(h+d_c)}\right)r\right]\tan^2\alpha$$
$$+\left[2\tan\phi(\tan\delta + \tan\phi) - \left(\frac{4V}{\gamma(h^2 - d_c^2)}\right)\tan\beta(1+\tan^2\phi) + \left(\frac{4c}{\gamma(h+d_c)}\right)s\right]\tan\alpha$$
$$+\left[t + \left(\frac{2V}{\gamma(h^2 - d_c^2)}\right)\tan^2\beta(1+\tan^2\phi) + \left(\frac{2c}{\gamma(h+d_c)}\right)r\right] = 0$$

Denoting the coefficient of $\tan^2\alpha$ as $-A$, the coefficient of $\tan\alpha$ as AC_1, and the constant as AC_2 we have the following quadratic equation:

$$- A \tan^2 \alpha + AC_1 \tan \alpha + AC_2 = 0$$

Dividing this by - A gives:

$$\tan^2 \alpha - C_1 \tan \alpha - C_2 = 0$$

The solution for tan α then is:

$$\tan \alpha = \frac{C_1 + \sqrt{C_1^2 + 4 C_2}}{2}$$

alternately

$$\alpha = \tan^{-1} \left(\frac{C_1 + \sqrt{C_1^2 + 4 C_2}}{2} \right)$$

where

$$A = \tan \phi + \tan \delta - \left(\frac{2V}{\gamma (h^2 - d_c^2)} \right) (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma (h + d_c)} \right) r$$

$$C_1 = \frac{2 \tan \phi (\tan \delta + \tan \phi) - \left(\frac{4V}{\gamma (h^2 - d_c^2)} \right) \tan \beta (1 + \tan^2 \phi) + \left(\frac{4c}{\gamma (h + d_c)} \right) s}{A}$$

$$C_2 = \frac{t + \left(\frac{2V}{\gamma (h^2 - d_c^2)} \right) \tan^2 \beta (1 + \tan^2 \phi) + \left(\frac{2c}{\gamma (h + d_c)} \right) r}{A}$$